Perturbation methods, algebra and nonlinear vibrations

C.-H. Lamarque[1], **A. Ture Savadkoohi**[1]

[1]Univ Lyon, ENTPE, LTDS UMR CNRS 5513
Vaulx-en-Velin, France
lamarque@entpe.fr, alireza.turesavadkoohi@entpe.fr

Abstract A technique based on exploiting Gröbner bases [1] of multivariable polynomials for detection of periodic solutions is introduced. It is based on generating Gröbner bases and verifying the possibility of determining/defining polynomials of approximated L^2-norms of solutions which belong to ideals generated by the Gröbner bases.

To track periodic solutions of nonlinear (quite often polynomial nonlinearities) smooth second order differential systems (i.e. nonlinear vibrations of nonlinear mechanical systems) perturbation methods have been developed (KBM averaging, Normal Form, Multiple scales, etc.: [2–19]). These approximated methods lead to algebraic polynomial equations. One can obtain approximations of periodic solutions by solving systems of algebraic equations with unknown coefficients (coefficients of truncated Fourier series for examples) and given parameters. Numerical techniques are used to solve these algebraic equations at given parameters' values such as Newton-Raphson methods, or continuation methods [20, 21] if solutions are tracked versus one (or several, which is not usually the case in practice) parameter(s). Nevertheless finding all solutions, and especially the isolated branches of solutions is always a challenge. For systems with k degrees of freedom (dof), $k \geq 1$, these perturbation methods tends to provide approximated values of coefficients of truncated Fourier series and then to provide frequency-response curves, finally the response corresponds to an approximation of L^2 norm of a periodic response of each dof. We present an approach based on using algebra methods. The main idea is to exploit Gröbner bases of multivariate polynomials. Contrary to the approach of Grolet and Thouverez [22], we do not try to obtain a parametrization of the nonlinear vibrations versus one particular variable which satisfies a polynomial equation. Here, the main idea is to test the belonging of a polynomial of the approximated L^2-norm to the ideal generated by the set of polynomial equations issuing from the analytical approximated response. We consider the cases of single dof systems. Let us assume that a perturbation/approximated method provides N polynomial equations. Then, a general algorithm is described to generate a Gröbner bases and to test the possibility to constraint a polynomial (to be determined) of the approximated L^2-norm of the solution of single dof systems to belong to ideals generated by the Gröbner basis. This approach is presented via some simple examples (Duffing oscillator at first) treated by Harmonic Balance method as an example of the perturbation method. Then, we explain how to extend the method to two dof systems. It is enough to understand possible generalization to n dof systems. The method could be extended to non polynomial nonlinearities. Limitations/potential of the method are discussed. Moreover, open questions and perspectives will be given.

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