Stationary and Non-stationary Resonant Dynamics of the Finite Chain of Weakly Coupled Pendula

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Abstract We discuss new phenomena of energy localization and transition to chaos in the finite system of coupled pendula without any restrictions on the amplitudes of oscillations. We propose a new approach to the problem based on the recently developed Limiting Phase Trajectory (LPT) concept in combination with a semi-inverse method. The analytic predictions of the conditions providing transition to energy localization are confirmed by numerical simulation.

The system of the coupled pendula is the wide-expanded model in the various field of science [1, 2]. We consider the nonlinear non-stationary dynamics of the sine-lattice [3], which is very useful in some fields of the polymer physics and biophysics. We start from the Hamilton function of the discrete system of coupled pendula with the harmonic-type bonds:

$$H = \sum_{j=1}^{N} \left(\frac{1}{2} \left(\frac{d\varphi_j}{dt} \right)^2 + (1 - \cos(\varphi_{j+1} - \varphi_j)) + \sigma \left(1 - \cos\varphi_j\right) \right), \tag{1}$$

where φ_j is the displacement of the j-th pendulum from its equilibrium position and N is the number of pendula. The dimensionless time t is normalized by the coupling constant while the "gravity" constant σ can be changed in the accordance of the concrete problem.

It can be shown [4] that the non-stationary dynamics of the system under consideration can be studied in the terms of the complex variables $\Psi_j = (1/\sqrt{2\omega})(\omega\varphi_j + id\varphi_j/dt)$. In such a case, the energy of the nonlinear normal mode $\Psi_j = \psi_j \exp(-i\omega t)$ can be written in the form

$$H_r = \sum_{j=1}^{N} \frac{\omega}{2}|\psi_j|^2 + \left(1 - J_0\left(\sqrt{\frac{2}{\omega}}|\psi_{j+1} - \psi_j|\right)\right) + \sigma\left(1 - J_0\left(\sqrt{\frac{2}{\omega}}|\psi_j|\right)\right), \tag{2}$$

where J_0 - the Bessel function of zero order and the eigen frequency of the nonlinear normal mode with the wave number κ can be written as follows

$$\omega^2 = \frac{2}{Q}\left(2J_1\left(2Q\sin\frac{\kappa}{2}\right)\sin\frac{\kappa}{2} + \sigma J_1(Q)\right). \tag{3}$$

Here, the amplitude Q of the oscillations are related to the modulus of the complex variable $|\psi_j| = \sqrt{\frac{\omega}{2}}Q$.

Equation (2) with $\psi_j = \chi \exp(i\kappa j)$ can be considered as the Hamilton function corresponding to the non-stationary dynamics if the amplitude χ is changed at the scale, which is essentially larger than the period of the mode ($T = 2\pi/\omega$). In particular, this occurs when two mode with close wave numbers are excited simultaneously. In such a case, the equations of motion can be obtained accordingly the rule:

$$i\frac{\psi_j}{d\tau} = -\frac{\partial H_r}{\partial \psi_j^*} \tag{4}$$

(the asterisk denotes the complex conjugate function).

It can be shown that equations (4) admit the additional integral of motion $X = \frac{1}{N}\sum_j |\psi_j|^2$. It was shown in [4–6] that the existence of integral X is extremely useful for the analysis of the nonlinear normal modes (NNMs) interaction. Near the edges of the spectrum (3) this process leads to the separation of the chain onto two domains, which are differed by the energy concentration. So we can introduce the "domain variables"

$$\chi_1(\tau) = \frac{1}{\sqrt{2N}}\sum_j \psi_j(\tau)\left(1+\cos(\kappa j + \frac{\pi}{4})\right); \quad \chi_2(\tau) = \frac{1}{\sqrt{2N}}\sum_j \psi_j(\tau)\left(1-\cos(\kappa j + \frac{\pi}{4})\right).$$

Taking into account integral X one should write the domain variables in the form $\chi_1 = \sqrt{X}\cos\theta\exp(-i\Delta/2), \chi_1 = \sqrt{X}\sin\theta\exp(i\Delta/2)$ and express the domain occupation via the value $R = |\chi_1|^2 - |\chi_2|^2 = X\cos 2\theta$. The accurate analysis of hamiltonian (3) on the phase plane Δ, R shows that there are two threshold values of integral X (see fig. 1(a-c)). Before the first threshold the modes are stable and the periodic redistribution of the energy between domains occurs. If X exceeds the first threshold one of the modes losses its stability, but the energy migration is still possible. Finally, above the second threshold the phase trajectories starting at $R = -1$ can not achieve the value $R = 1$ and vise versa. It means that the energy putting into one part of the chain can not be redistributed along it.

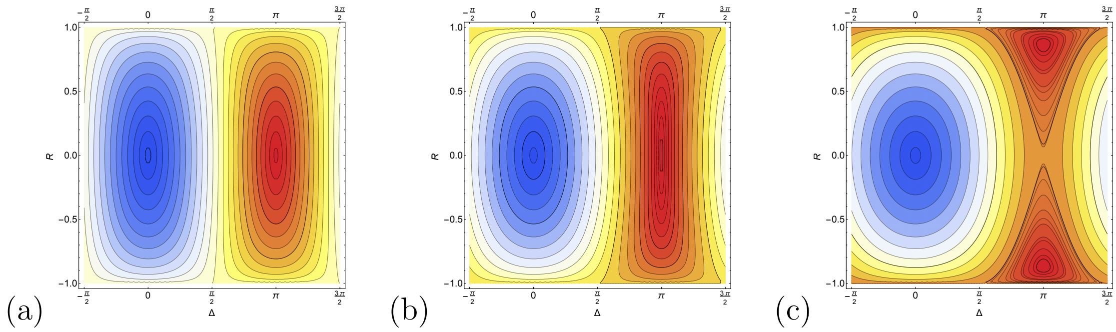

Figure 1: Phase portraits (2) in the terms of variables R and Δ at different oscillation amplitudes $Q = \pi/10$(a). $Q = 2\pi/10$ (b), $Q = 3.2\pi/10$ (c). $\sigma = 1$, $N = 32$.

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