A Taylor series based continuation method for equilibrium, periodic, quasi-periodic and transient solutions of dynamical systems

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Abstract. This paper emphasizes how a quadratic rewriting of ordinary differential equations (ODE) allows different types of solutions to be easily continued by asymptotic numerical method (ANM). The focus will be especially on the continuation of quasi-periodic steady state solutions and transient solutions. A toy model of saxophone is studied in detail to illustrate the methods presented.

Introduction. The ANM relies on a high-order Taylor series representation of the solution-branch. This technique has already proven its efficiency for a lot of applications in engineering, mechanics or acoustics for example. As opposed to standard predictor-corrector algorithm, the high order prediction of ANM series does not need a correction step in most cases. While some implementations relying on automatic differentiation do exist the choice is made here to work with a quadratic framework. A generic implementation of this latest approach which minimizes problem-dependent implementation has been developed [3, 4]. A simplified scheme is represented in figure 1. It is based on the numerical continuation of algebraic systems of the form

$$R(V) = 0, \qquad \text{where } V \in \mathbb{R}^{n+1} \text{ and } R(V) \in \mathbb{R}^n \text{ is analytic} \tag{1}$$

This system is always written in a quadratic format as a prerequisite of the method. This formalism is not a constraint that we suffer but a choice that allows to treat a very wide range of problems as shown in [3].

The quadratic framework. The quadratic rewriting of the system is a key point of our approach since it allows to compute the terms in the development of the series explicitly and very efficiently (see [3] for details). Here, the focus is on ODE :

$$\dot{X} = F(X, \lambda, t), \qquad \text{where } X \text{ and } F \in \mathbb{R}^n \text{ and } \lambda \in \mathbb{R} \tag{2}$$

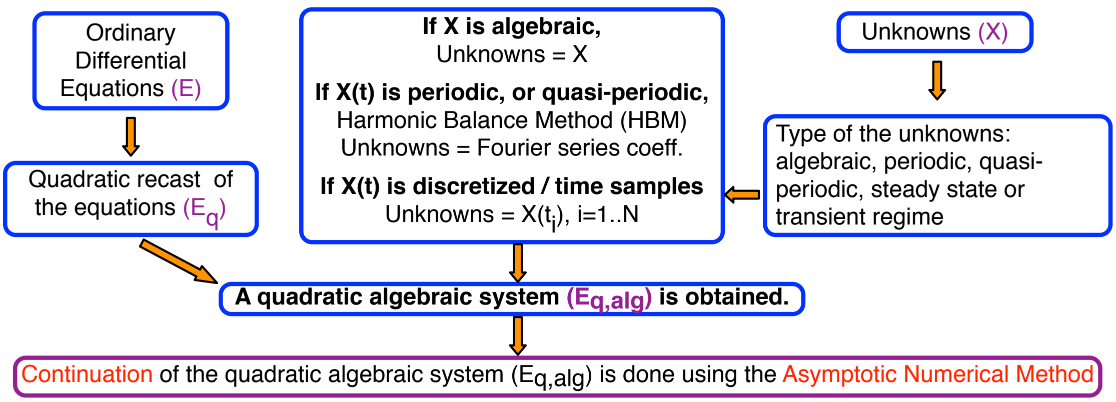

Figure 1: Different types of ODE solutions that can be easily continued by Asymptotic Numerical Method using quadratic rewriting.

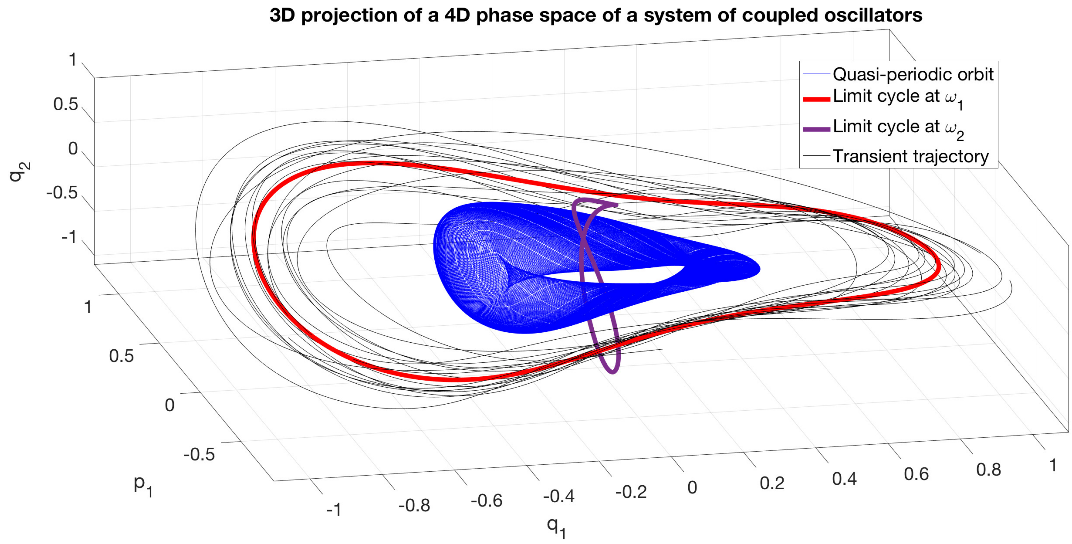

Figure 2: The two limit cycles and a quasi-periodic orbit of a system of a toy model of saxophone [4] obtained with our method. A transient that goes to one of the limit cycles is also represented.

Equ. (2) is solved using a quadratic recast. The equilibrium of ODEs and their stability can be classically treated by solving $F(X, \lambda, t) = 0$ and is not detailed. The periodic solutions and their stability is addressed with the harmonic balance method [4] and Hill's method [1]. The focus is on quasi-periodic and transient solutions.

A quasi-periodic solution can be sought for under the form of a truncated double Fourier series

$$X(t) = \sum_{k_1=-H}^{H} \sum_{k_2=-H}^{H} X_{k_1,k_2} e^{j(k_1\omega_1+k_2\omega_2)t}, \qquad \text{where } X_{k_1,k_2} \in \mathbb{C},\ \omega_1, \omega_2 \in \mathbb{R}_+^\star. \tag{3}$$

Replacing $X(t)$ by (3) in equ (2) and deriving a quadratic algebraic system from the recast of equ. (2) can be automatized [2]. Continuation is then possible on unknowns X_{k_1,k_2}, ω_1 and possibly ω_2 if the system is autonomous.

To continue transient regimes of ODEs, the solution X is discretized on time samples

$$X(t_i), 1 \leq i \leq N, N \in \mathbb{N}^\star. \tag{4}$$

Then, the quadratic rewriting of equ. (2) is written on each time step so that the equations are automatically quadratic. A wide range of discretization schemes can be used : Runge-Kutta, Euler, Newmark, finite-differences*etc.*.. Once the initial value is specified, continuation is possible on unknowns $X(t_i)$ and possibly t_i for schemes with auto-adaptive time steps.

Conclusion. The details of the methods discussed above are available in the journal article [4]. The method requires very few effort to switch between the different types of solutions since the original ODE is rewritten quadratically by the user once and for all. It can also be applied to implicit differential-algebraic systems with time-delay or fractional order derivatives. An implementation of this approach is freely available online on a dedicated website http://manlab.lma.cnrs-mrs.fr/.

References

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