Low-dimensional Nonlinear Modes computed with PGD/HBM and Reduced Nonlinear Modal Synthesis for Forced Responses

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Abstract This work proposes an algorithm allowing to perform a fast and light computation of branches of damped Nonlinear Normal Modes (dNNMs). Based on a previous work about undamped NNMs (uNNMs), it couples Proper Generalized Decomposition (PGD) features, harmonic balance and prediction-correction continuation schemes. After recalling the main contributions of the method applied on an example with cubic nonlinearities, the issue of a reduced nonlinear modal synthesis is briefly addressed.

The differential equations governing the motion of a nonlinear dynamical system can usually take the following form after a spatial discretization, where the nonlinear efforts $\mathbf{f}_{\text{nl}}(\mathbf{x}, \dot{\mathbf{x}})$ are separated from the linear ones:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{\text{nl}}(\mathbf{x}(t), \mathbf{x}(t)) = \mathbf{f}_e(t) \quad (\text{dim.: } N) \tag{1}$$

Assuming the Rosenberg's framework [4], an undamped NNM is a set of limit cycles of Eq. (1) from which dissipative and external forcing terms are put to zero. Shaw and Pierre extended this first definition of NNMs to the case of dissipative systems: a (damped) NNM is a two-dimensional invariant manifold in the phase space [5]. Hence, one wants to obtain a dNNM by computing a set of pseudo-periodic solutions of Eq. (1) with $\mathbf{f}_e(t) = \mathbf{0}$.

A frequential reduced algorithm coupling a PGD approach with an harmonic balance method (HBM) is implemented to make a quick and compact dNNM computation. The first step of the PGD process is separating the variables, here space and time:

$$\mathbf{x}(t) \approx \sum_{j=1}^{m} \mathbf{p}_j q_j(t) \Leftrightarrow \mathbf{x}(t) \approx \mathbf{P}\mathbf{q}(t) \text{ with } \mathbf{P} = [\mathbf{p}_1, \dots, \mathbf{p}_m] \tag{2}$$

$m \ll N$ is a positive integerand denotes the number of PGD modes $(\mathbf{p}_j, q_j(t))$ used for $\mathbf{x}$ decomposition. $\mathbf{P}$ is the $(N \times m)$-sized matrix of the m PGD mode shapes $\mathbf{p}_j$, and $\mathbf{q}(t)$ is the vector containing the time dependence of each PGD mode. Then a spatial subproblem $\mathcal{S}_m$ and a temporal subproblem $\mathcal{T}_m$ are defined from specific weak formulations [1]. The calculus, its notations and operators are detailed in [3]. Given the spatial matrix $\mathbf{P}$, $\mathcal{T}_m$ is a set of m ordinary differential equations of order 2. A complex HBM [2] is then implemented to obtain another nonlinear algebraic system for $\mathcal{T}_m$ from the following solution form:

$$\mathbf{q}(t) = \frac{\mathbf{a}_0}{\sqrt{2}} + \sum_{k=1}^{H} e^{-k\beta t}[\mathbf{a}_k \cos(k\omega t) + \mathbf{b}_k \sin(k\omega t)] \tag{3}$$

β-dependant linear terms can easily be separated from the classic undamped HBM matrices. Given the temporal part $\mathbf{q}(t)$, the spatial subproblem $\mathcal{S}_m$ is a $N \times m$ nonlinear algebraic system which takes into account the damping ratio β into the definition of its operators. The PGD/HBM solver is obtained by integrating $\mathcal{S}_m$ and $\mathcal{T}_m$ into an alternated directions solver. This PGD/HBM solver is eventually embedded into a continuation scheme as a corrector in order to build the dNNM branch. The choice of predictor is left to the user.

The first point of the branch is described with only one PGD mode which contains the shape and the frequency of the underlying linear damped mode with a null amplitude. When the error criterion can no longer be met through the continuation, the size of the description m is incremented and some new spatial and temporal information is added to the PGD description. Unlike Grolet and Thouverez [1] who initialized $\mathbf{q}(t)$ with random values, we propose to process $\mathcal{T}_m$ first based on the shape on the next damped linear modes. The full algorithm adds new modal data on the fly, only when it is necessary with respect to the error criterion.

It should be noted that two variants have been implemented: oPGD (optimized PGD) recomputes the whole $\mathbf{P}$ matrix at each point of the branch whereas pPGD (progressive PGD) only computes the last shape $\mathbf{p}_{m+1}$ when it is introduced. Although oPGD generally needs less PGD modes than pPGD, the computational cost is higher.

The method is here applied on a cantilever beam with a cubic spring at its free end investigated in [3]. A 1% modal damping is added on each linear mode. The Frequency-Energy Plot given on Fig. 1 illustrates the hardening effect of the cubic nonlinearity while the conservative mechanical energy grows and the damping-energy dependence of the NNM.

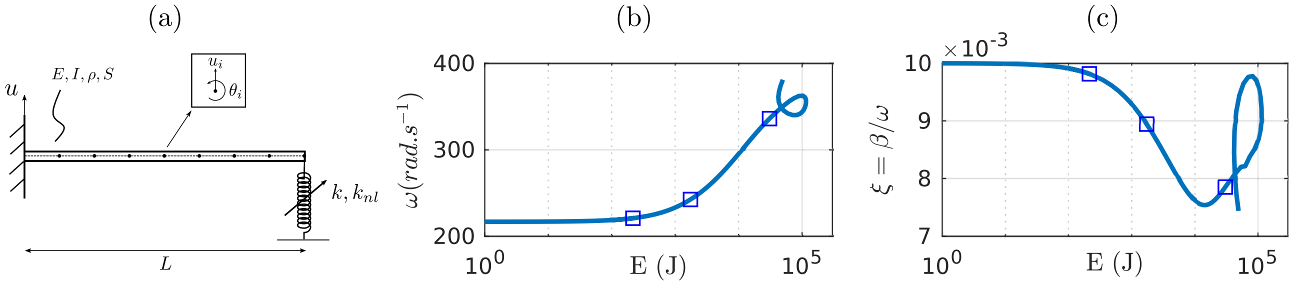

Figure 1: (a) Schematic diagram of beam+cubic spring (b) Main branch of dNNM1 (c) Damping ratio with respect to energy. Squares: Solution points where a PGD mode is added.

Eventually, this compact description of NNMs can be embedded into a reduced modal synthesis solver in order to quickly build Frequency Response Functions by looking for $\mathbf{x}$ solution of Eq. (1) as follows:

$$\mathbf{x}(t) = \mathbf{P}(s)\left(\frac{\mathbf{a}_0(s)}{\sqrt{2}} + \sum_{k=1}^{H}[\mathbf{a}_k(s)\cos\left(k(\omega t+\phi)\right) + \mathbf{b}_k(s)\sin\left(k(\omega t+\phi)\right)]\right) \tag{4}$$

where s is an index for the dNNM branch and ϕ is the phase of the response. Although this decomposition is slightly different of the one given in [2], the two equations used to solve for s and ϕ are similar.

References

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