Phase driven modal synthesis for forced response evaluation

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Abstract A new definition is proposed for the Nonlinear Normal Modes, close to the one developed by Bellizzi & Bouc [1]. Theses NNMs are the used to evaluate the forced responses using a modal phase parametrization rather than the classical forcing frequency parametrization.

The basic dynamic equation considered for nonlinear dynamics writes

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} + \mathbf{f}_{\mathrm{nl}}(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{f}_{\mathrm{e}}(t) \tag{1}$$

where $\mathbf{f}_{\mathrm{nl}}$ gathers nonlinear forces while $\mathbf{f}_{\mathrm{e}}$ denotes a periodic external forcing.

Damped nonlinear normal modes (dNNMs) are the solutions of Eq. (1) when the forcing $\mathbf{f}_{\mathrm{e}}$ is nullified [3]. Several methods to compute these solutions where proposed. The one exposed and used here is close to the amplitude and phase parameterization described by Bellizzi and Bouc [1]. Displacements $\mathbf{u}$ and velocities $\mathbf{v}$ have the same dependency to an amplitude α and a dimensionless time τ than in [1] but the amplitude decay function η and the pseudo circular frequency ω only depend on amplitude here:

$$\mathbf{u}(t) = \alpha(t)\boldsymbol{\psi}^u(\alpha(t), \tau(t)),\ \mathbf{v}(t) = \alpha(t)\boldsymbol{\psi}^v(\alpha(t), \tau(t)),\ \dot{\alpha}(t) = \eta(\alpha(t))\alpha(t),\ \dot{\tau}(t) = \omega(\alpha(t)) \tag{2}$$

Once injected in Eq. (1), and adding $\mathbf{v} = \dot{\mathbf{u}}$ condition leads to

$$\alpha\boldsymbol{\psi}^v(\alpha, \tau) = \eta(\alpha)\alpha\boldsymbol{\psi}^u(\alpha, \tau) + \alpha\left(D_\alpha\boldsymbol{\psi}^u(\alpha, \tau)\eta(\alpha)\alpha + D_\tau\boldsymbol{\psi}^u(\alpha, \tau)\omega(\alpha)\right) \tag{3a}$$

$$\begin{aligned}\mathbf{M}\left(\eta(\alpha)\alpha\boldsymbol{\psi}^v(\alpha, \tau) + \alpha\left(D_\alpha\boldsymbol{\psi}^v(\alpha, \tau)\eta(\alpha)\alpha + D_\tau\boldsymbol{\psi}^v(\alpha, \tau)\omega(\alpha)\right)\right) + &\\ \mathbf{C}\left(\alpha\boldsymbol{\psi}^v(\alpha, \tau)\right) + \mathbf{K}\left(\alpha\boldsymbol{\psi}^u(\alpha, \tau)\right) + \mathbf{f}_{\mathrm{nl}}(\alpha\boldsymbol{\psi}^u(\alpha, \tau), \alpha\boldsymbol{\psi}^v(\alpha, \tau)) &= \mathbf{0}\end{aligned} \tag{3b}$$

Instead of seeking for the various quantities as a power series in α and a Fourier series in τ which leads to a very large system of equations, a "point-by-point" approach is preferred in the α dimension: a branch is defined by successive points gathering $\alpha^{(i)}$ (modal amplitude), $\mathbf{Q}^{u^{(i)}}$ (Fourier coefficients for $\boldsymbol{\psi}^{u^{(i)}}$), $\mathbf{Q}^{v^{(i)}}$ (Fourier coefficients for $\boldsymbol{\psi}^{v^{(i)}}$), $\eta^{(i)}$ (modal amplitude decay function) and $\omega^{(i)}$ (modal circular frequency). While $D_\tau\bullet = \partial\bullet/\partial\tau$ quantities can be evaluated exactly via Fourier series derivation, $D_\alpha\bullet = \partial\bullet/\partial\alpha$ is evaluated using a linear interpolation between the previous and the current points. The two necessary normalization conditions are defined by

$$\boldsymbol{\psi}^u(\alpha, 0)^{\mathrm{T}}\mathbf{M}\boldsymbol{\psi}^u(\alpha, 0) + \boldsymbol{\psi}^u(\alpha, \pi/2)^{\mathrm{T}}\mathbf{M}\boldsymbol{\psi}^u(\alpha, \pi/2) = 1 \tag{4a}$$

$$\boldsymbol{\psi}^u(\alpha, 0)^{\mathrm{T}}\mathbf{M}\boldsymbol{\psi}^u(\alpha, \pi/2) = 0 \tag{4b}$$

Lastly, points on the branch are index by their (discrete) arclength $s^{(i)}$:

$$\begin{aligned}s^{(i)} = s^{(i-1)} + \Big(&\left(\alpha^{(i)} - \alpha^{(i-1)}\right)^2 + \left(\eta^{(i)} - \eta^{(i-1)}\right)^2 + \left(\omega^{(i)} - \omega^{(i-1)}\right)^2 \\ &+ \left|\left|\mathbf{Q}^{u^{(i)}} - \mathbf{Q}^{u^{(i-1)}}\right|\right|^2 + \left|\left|\mathbf{Q}^{v^{(i)}} - \mathbf{Q}^{v^{(i-1)}}\right|\right|^2\Big)^{1/2}\end{aligned} \tag{5}$$

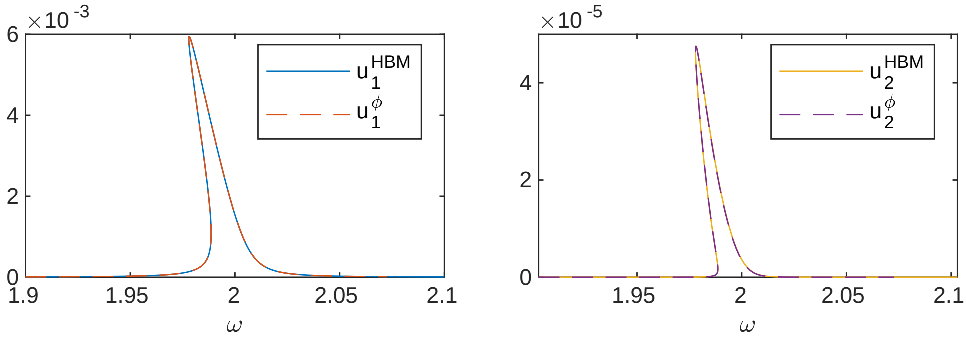

Figure 1: Illustration: modal synthesis around first mode for a 2-dofs system.

Once a dNNM is calculated, it offers a first understanding of the structure as well as a rough prediction of its behavior when forcing is introduced. It can also be used to compute the forced response effectively using modal synthesis.
Let us assume that $\mathbf{f}_\mathrm{e}(t) = \mathbf{f}_{\mathrm{e}_0} \cos(\omega t)$. Using a dimensionless time $\tau = \omega t$ and denoting $\mathbf{u}_\tau(\tau) = \mathbf{u}(t)$, $\bullet' = d \bullet / d\tau$, Eq. (1) becomes

$$\omega^2 \mathbf{M}\mathbf{u}''_\tau + \omega \mathbf{C}\mathbf{u}'_\tau + \mathbf{K}\mathbf{u}_\tau + \mathbf{f}_\mathrm{nl}(\mathbf{u}_\tau, \omega \mathbf{u}'_\tau) = \mathbf{f}_{\mathrm{e}_\tau}(\tau) \tag{6}$$

Then, $\mathbf{u}_\tau$ is naturally sought as

$$\mathbf{u}_\tau(\tau) = \tilde{\mathbf{u}}(s, \tau + \phi) \tag{7}$$

where the 2 unknowns are s which defines the location on the dNNM branch and ϕ, the phase with respect to the excitation as in the linear case.
Equations used to find these 2 unknowns are

$$\int_0^{2\pi} \mathbf{r}(\tau)\, \tilde{\mathbf{u}}(s, \tau + \phi)\, d\tau = 0 \text{ and } \int_0^{2\pi} \mathbf{r}(\tau)\, \left(\omega \tilde{\mathbf{u}}'(s, \tau + \phi)\right)\, d\tau = 0 \tag{8}$$

with $\mathbf{r}(\tau)$ being the residue of the dynamical equation (6):

$$\mathbf{r}(\tau) = \omega^2 \mathbf{M}\mathbf{u}''_\tau + \omega \mathbf{C}\mathbf{u}'_\tau + \mathbf{K}\mathbf{u}_\tau + \mathbf{f}_\mathrm{nl}(\mathbf{u}_\tau, \omega \mathbf{u}'_\tau) - \mathbf{f}_{\mathrm{e}_\tau}(\tau) \tag{9}$$

This system can be solved using any continuation method in the variables ω, s, ϕ.

Another approach is to consider that, as in the linear case, ϕ will vary from 0 to $-\pi$ with a continuous decrease along the frequency function response (FRF). Hence, the FRF can be computed by solving for ω and s only for discrete values of $\phi \in]-\pi, 0]$ avoiding the use of a continuation scheme. This approach was applied to compute the first mode and the FRF around this first mode for the 2-dofs example used by Touzé and Amabili [4] and return very accurate results as illustrated in Figure 1 for which reference results are HBM results with up to 5 harmonics. This phase parameterization can be very interesting in the stochastic case to link points of different realizations as explained in [2] for the linear case.

References

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